Intro.

Many times during this course we start off with some intuition about interesting phenomena, we model them, giving them a representation that is friendly and manageable, and very often we quantify the phenomena. Some math is of great importance when it comes to representing and measuring.

Numbers.

If quantifying is part of what we do, we need to count. What kind of numbers do we use?

$$0, 1, 2, \dots$$

the most *natural* kind of numbers you can think of for counting, and their opposite

$$\dots, -2, -1$$

and the ratio of any such numbers, for example

$$\frac{1}{2}, \frac{3}{4}, \frac{3}{5}, \frac{-2}{3} \dots$$

So what is the rule for multiplying and dividing ratios? Let's consider these examples.

$$\frac{3}{5} \div \frac{-2}{3} = \frac{3}{5}\frac{-3}{2} = \frac{-3 \cdot 3}{5 \cdot 2} = \frac{-9}{10}$$

$$\frac{3 \cdot \frac{2}{3}}{\frac{-1}{3}} \div 5 = 3 \cdot \frac{2}{3} \div \frac{-1}{3} \div 5 = 3 \cdot \frac{2}{3} \cdot (-3) \cdot \frac{1}{5} = \frac{3 \cdot 2 \cdot (-3)}{3 \cdot 5} = \frac{-18}{15}$$

Functions.

We often look to understand what is the relation among some variables. Is the value of a variable dependent on the value of some other variable? For example, how much we consume can be dependent on how much we earn, the amount of energy we use up depends on how many light bulbs are switched on, and so forth. That is the intuitive idea of what a function is.

Let's consider this last case. The amount of energy consumed e *is a function* l, the number of light bulbs that are on,

$$e = f(l),$$

and if we know the nature of this function we can write it down explicitly; for example, if we know that each light bulb consumes 2 units of energy,

$$e = 2l.$$

Another example, "My amount of happiness is a function of the number of slices of pizza I eat, and even if I don't eat any, I am still pretty happy" could be represented as:

$$h = 2 + s.$$

Tables.

How do we approach the problem of understanding how the function we are considering looks like. We can tabulate it. A table is in fact an organized list of items and values. Let's consider again: the amount of energy consumed is a function of the number of light bulbs that are on:

$$e = 2l.$$

And here is the table.

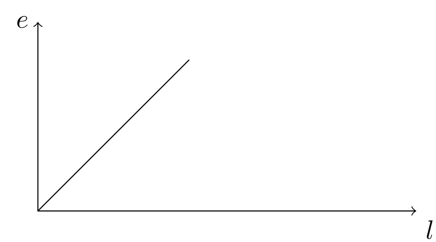
Number of light bulbs, l	Total energy, e
0	0
1	2
2	4
3	6
4	8
$\vdots$	$\vdots$

But we cannot see patterns on a table. We need a tool that is more intuitive and friendly: a graph.

Graphs.

A graph is a place where we can draw quantities. So if we have a pair of values we can represent them in two dimensions, and this is in fact the only type of graph we will deal with.

So if we have a bunch of pairs of values, we wish to represent them where it makes sense to do so. Hence, we need the following elements: axes, labels of the axis, optionally unit of measure of the variables at hand.



Explicit and implicit function, line.

The function is written down equating the two sides as in

$$h = 2 + s.$$

In this case one clearly sees that h is a function of s, as the function is written *explicitly*, and one variable is *independent*, while the other is *dependent*, as it derives its value from the value of the independent variable. Here the same function is written implicitly:

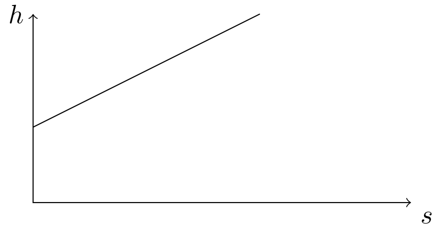
$$h - 2 - s = 0.$$

One cannot plot a function that is in its implicit form: the function has to be put in its explicit form.

Now let's draw

$$h = 2 + s.$$

s	h
0	2
1	3
2	4
3	5
4	6
⋮	⋮



Now both these functions are plotted as lines, and therefore we call them linear functions of simply lines. Why are they lines? Because the variables vary proportionally in a fixed way. For example, with

$$h = 2 + s$$

increasing the independent variable s of one unit, always yields one additional unit of the dependent variable.

Some features of a line are very important: intercept and slope; intercept is the value of the dependent variable, when the independent variable is zero; the slope is the rise over the run, that is, variation in the dependent variable over variation in the independent variable.

Here we introduce a small piece of notation: Δ stands for variation, so that $\Delta x = x_1 - x_0$, where the convention is to take final value minus initial value. Then, we can write the slope as

$$\frac{\Delta y}{\Delta x}.$$

So, what are the slope and the intercept of $h = 2 + s$ and $e = 2l$?

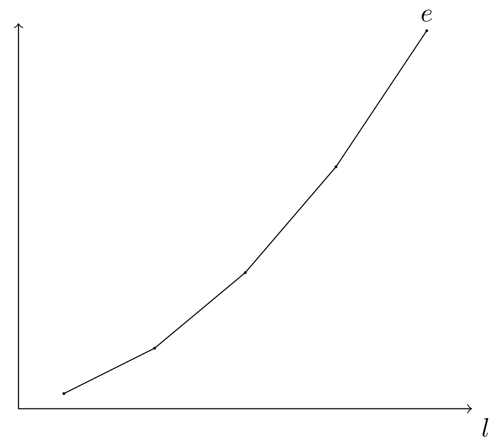
Notice that the slope of a line is unique, it does not depend on what points have been chosen to calculate it.

Total, margin, average.

Now let's consider this table.

Number of light bulbs, l	Total energy, e
1	1
2	4
3	9
4	16
5	25
⋮	⋮

And let's graph it.



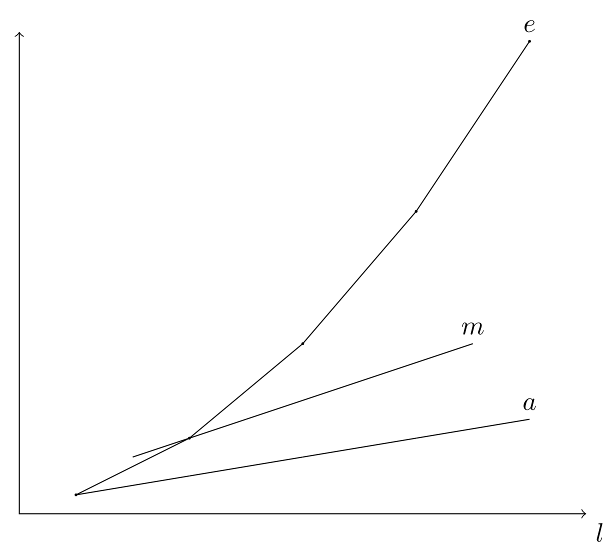
We easily see that there is no single line that goes through all the points: this bunch of points that we got from the table and plotted, cannot have been generated by a linear function.

Total quantities are not the only thing we care about. In fact we often want to know averages and margins. The concept of margin is *the variation of the dependent variable due to a unitary change in the independent variable.* The average is obtained as *the ratio of the total quantity of the dependent variable over the quantity of the independent variable*:

$$\frac{y}{x}.$$

Let's tabulate and plot these quantities in our example.

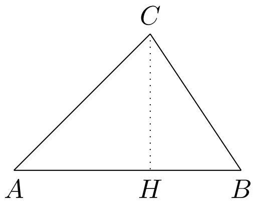
l	e	Average, a	Margin, m
1	1	1	3
2	4	2	5
3	9	3	7
4	16	4	9
5	25	5	
⋮	⋮	⋮	⋮



It is interesting to notice the fact that the margin drives the average.

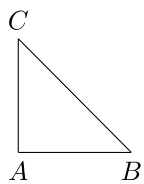
Area of a triangle, area under a curve.

We need to know how to compute the area of a triangle.



The rule is, base times height over two

$$\text{area} = \frac{b \cdot h}{2} = \frac{AB \cdot HC}{2}.$$



If the triangle is a right triangle we can take the product of the legs divided by two

$$\text{area} = \frac{l_1 \cdot l_2}{2} = \frac{AB \cdot AC}{2}.$$

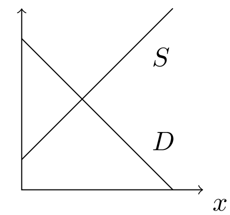
This also allows us to calculate the area under a curve, or between curves, if we can manage to decompose it into rectangles and triangles.

As an example, we consider the following lines,

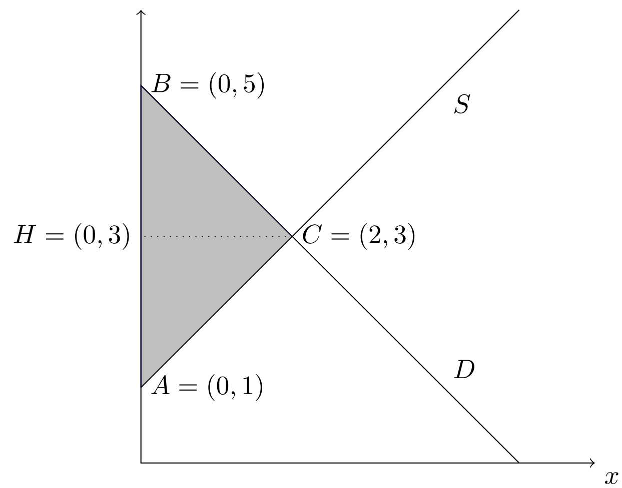
$$S : \quad y - x - 1 = 0,$$

$$D : \quad y + x - 5 = 0.$$

We plot them (if it helps tabulate them first)



and now we compute the area between the two lines, to the left of their intersection.



$$area = b \cdot h/2 = AB \cdot HC/2 = 4 \cdot 2/2 = 4.$$